STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

MR 39 a r

The Numerical Transformation of Slowly Convergent Series by Methods of Comparison
Part II

P. Wynn

(Chiffres)

1962

CHIFFRES - 2 - 1962.

The Numerical Transformation of Slowly Convergent Series by Methods of Comparison

Part II *

by
P. WYNN,
Centre mathématique Amsterdam.

Théorie de la convergence et généralisations de la transformation :

$$\sum_{s=0}^{\infty} c_s v_s x^s \sim \sum_{s=0}^{m-1} c_s v_s x^s + \sum_{s=0}^{\infty} x^{m+s} \overset{(s)}{\Phi}_m(x) \Delta^s v_m \qquad a)$$

où

$$\Phi_m(x) \sim \sum_{s=0}^{\infty} c_{m+s} x^s \qquad b)$$

sont données. Des exemples montrent comment cette technique de comparaison peut être accélérée davantage par l'application de l'ε-algorithme.

Generalisations of, and a convergence theory for, the transformation
a)
where
b)
are given. Illustrations are given of how this comparison technique may further be accelerated by application of the ε-algorithm.

Gegeben sind die Konvergenztheorie und die Verallgemeinerung der Transformation
a)
wo
b)
Beispiele zeigen, wie dieses Vergleichsverfahren noch weiter beschleunigt werden kann durch die Verwendung des ε-algorithmus.

* Part I of this paper appeared in *Chiffres*, Vol. 4, 1961, p. 177-210.

Автор дает теорию сходимости и обобщение преобразования :

а)

где б)

Примеры укаэывают каким обраэом эту технику сравнения можно еще более ускорить применением ε-алгорифма.

Second and Higher Order Versions of the Euler Guderman Transformation.

7-1. It will have been noticed from the preceding discussion and examples, that the success of the transformations depends to a large extent upon whether or not the quantities v_n are slowly varying functions of n. An artifice, the use of which may feasibly increase the efficiency of the transformations, is to use an expression of the form

$$v_n = f\,(n;\ \overset{(0)}{v_n},\ \overset{(1)}{v_n},\ \ldots,\ \overset{(k)}{v_n}) \tag{7-1-1}$$

where the variation with n of the quantities $\overset{(0)}{v_n},\ \overset{(1)}{v_n},\ \ldots,\ \overset{(k)}{v_n}$ is slower than that of v_n. The differences $\Delta^s v_n$ may then, in suitable cases, be expressed in terms of the differences of the quantities $\overset{(0)}{v_n},\ \overset{(1)}{v_n},\ \ldots,\ \overset{(k)}{v_n}$ and it may well transpire that the rearranged form of the Euler-Gudermann transformation is more suitable for numerical computation than the original version.

7-2. An example will assist in clarifying the discussion. Suppose that, either by reference to an explicit formula for v_n, or by reasoning based upon a difference equation satisfied by v_n, it is possible to show that

$$v_n = O\left(\sum_{s=-p}^{+q} \overset{(s)}{V}\, n^{-s}\right) \tag{7-2-1}$$

where the coefficients $\overset{(s)}{V}$ are constant. Then, writing

$$v_n = \sum_{s=-p}^{+q} \overset{(s)}{V}\, n^{-s} + \overset{(q+1)}{v_n}\, n^{-q-1} \tag{7-2-2}$$

it may transpire that the variation of $\overset{(q+1)}{v_n}$ with n is less than that of v_n. The Euler-Gudermann transformation may then be modified to read

$$\sum_{r=0}^{\infty} c_r v_r x^r = c_0 v_0 + \sum_{r=1}^{\infty} c_r \left\{ \sum_{s=-p}^{+q} v^{(s)} r^{-s} + v_r^{(q+1)} v^{-q-1} \right\} x^r$$
$$= c_0 (v_0 - v^{(0)}) + v^{(0)} \phi_0(x) + x \sum_{r=1}^{q} V^{(r)}\, {}_r\phi_0(x) + x \sum_{r=1}^{p} V^{(-r)}\, {}_{-r}\phi_0(x)$$
$$+ x \left\{ {}_{q+1}\phi_0(x) v_1^{(q+1)} + \frac{x}{1!}\, {}_{q+1}\phi_0'(x) \Delta v_1^{(q+1)} + \frac{x^2}{2!}\, {}_{q+1}\phi_0''(x) \Delta^2 v_1^{(q+1)} + \dots \right\} \quad (7.2.3)$$

where :

$${}_r\phi_0(x) = \sum_{n=1}^{\infty} c_{n+1} n^{-r} x^{n-1} \qquad r = -p, -p+1, \dots, -1, 1, \dots, q, q+1 \quad (7.2.4)$$

If the quantities v_r satisfy a linear difference equation, then such an equation for $\overset{(q+1)}{v_r}$ may easily be constructed.

The foregoing analysis clearly permits at least the formal possibility of substituting

$$v_n = \sum_{s=-p}^{q+1} \overset{(s)}{v_n}\, n^{-s} \qquad (7\text{-}2\text{-}5)$$

where all the coefficients $\overset{(s)}{v_n}$ $s = -p, -p+1, \dots, q+1$ are slowly varying functions of n. This involves the further difficulties of obtaining initial values $\overset{(s)}{v_0}$ and obtaining a law of formation for the quantities $\overset{(s)}{v_n}$. It may be possible to overcome the first by interpretation of an explicit formula and the second by rearrangement of a linear difference equation in congruent powers of n. The Euler-Gudermann transformation then evolves to the form

$$\sum_{s=0}^{\infty} c_s v_s x^s \sim c_0 v_0 + \sum_{s=0}^{\infty} \sum_{r=-p}^{q+1} {}_r\overset{(s)}{u_0}\, \overset{s}{\Delta}\, \overset{(r)}{v_0} \qquad (7\text{-}2\text{-}6)$$

where, in addition to equations (7-2-4),

$${}_0\Phi_0(x) = \sum_{s=0}^{\infty} c_{s+1} x^s \qquad (7\text{-}2\text{-}7)$$

It is of interest to derive recursion systems between the quantities

$${}_r\overset{(s)}{u_0} = \frac{x^s}{s!}\, {}_r\overset{(s)}{\Phi_0}(x) \quad s = 0, 1, \dots ;\ r = -p, -p+1, \dots, q+1 \qquad (7\text{-}2\text{-}8)$$

The quantities ${}_r u_o = {}_r\Phi_o(x)$ $r = 1,2, \ldots$ must be determined by successive integration. In particular

$$ {}_0\Phi_0\ (x) = x^{-1}\ (\Phi_0\ (x) - c_0) \qquad (7\text{-}2\text{-}9) $$

and further

$$ {}_r\Phi_0\ (x) = x^{-1} \int_0^x {}_{r-1}\ \Phi_0\ (t)\ dt \qquad r = 1,2, \ldots \qquad (7\text{-}2\text{-}10) $$

For the function ${}_1\overset{(s)}{u_0}$ there obtain

$$ {}_1\overset{(1)}{u_0} = x^{-1}\ (\overset{(0)}{u_0} - c_0) - {}_1\overset{(0)}{u_0} \qquad (7\text{-}2\text{-}11) $$

and thereafter

$$ {}_1\overset{(s+1)}{u_0} = x^{-1}\ (s+1)^{-1} \left\{ \overset{(s+1)}{u_0} - x\ (2s+1)\, {}_1\overset{(s)}{u_0} - x\ {}_1\overset{(s-1)}{u_0} \right\} $$
$$ s = 1,2, \ldots \qquad (7\text{-}2\text{-}12) $$

and for the functions ${}_r\overset{(s)}{u_0}$ $r = 2,3, \ldots$

$$ {}_r\overset{(s+1)}{u_0} = \frac{{}_{r-1}\overset{(s)}{u_0}}{(s+1)} - {}_r\overset{(s)}{u_0} \qquad s = 0,1, \ldots \qquad (7\text{-}2\text{-}13) $$

For negative r the functions ${}_r\overset{(s)}{u_0}$ satisfy in turn

$$ {}_{-1}\overset{(s)}{u_0} = (s+1)\, x^{-1}\ \overset{(s+1)}{u_0} \qquad s = 0,1, \ldots \qquad (7\text{-}2\text{-}14) $$

and

$$ {}_{-r}\overset{(s)}{u_0} = (s+1) \left\{ {}_{-r+1}\overset{(s+1)}{u_0} + {}_{-r+1}\overset{(s)}{u_0} \right\} $$
$$ s = 0,1, \ldots\ ;\ r = 2,3, \ldots \qquad (7\text{-}2\text{-}15) $$

7-3. An auxiliary substitution, alternative to (7-2-1), which may in conjunction with linear difference equations, be easier to handle, is

$$ v_n = \sum_{s=1}^{p} n\,(n-1)\ldots(n-s+1)\ \overset{(s)}{\nabla\!\!|}_n + \sum_{s=1}^{q+1} \frac{\overset{(-s)}{\nabla\!\!|}_n}{n(n+1)\ldots(n+s-1)} + \overset{(0)}{\nabla\!\!|}_n $$
$$ n = 1,2, \ldots \qquad (7\text{-}3\text{-}1) $$

The Euler-Gudermann transformation then becomes

$$\sum_{s=0}^{\infty} c_s \, v_s \, x^s \sim c_0 \, v_0 + x \sum_{s=0}^{\infty} \sum_{r=-p}^{q+1} {}_r\mathrm{u}_0^{(s)} \Delta^s \, \nabla^{(r)}\!\big|_0 \qquad (7\text{-}3\text{-}2)$$

where

$$ {}_r\mathcal{U}_0^{(s)} = \frac{x^s}{s!} \, {}_r\Phi_0^{(s)}(x) \qquad (7.3.3)$$

$$ {}_r\Phi_0(x) = \frac{c_1}{1.2\ldots r} + \frac{c_2 x}{2.3\ldots(r+1)} + \frac{c_3 x^2}{3.4\ldots(r+2)} + \ldots \qquad (7.3.4)$$
$$r = 1, 2, \ldots$$

$$ {}_{-r}\Phi_0(x) = r(r-1)\ldots 1 \, c_r + (r+1) r (r-1) \ldots 2 \, c_{r+1} x + \qquad (7.3.5)$$
$$ + (r+2)(r+1) r \ldots 3 \, c_{r+2} x^2 + \ldots \qquad r = 1, 2, \ldots$$

and again

$$ {}_0\Phi_0(x) = \sum_{s=0}^{\infty} c_{s+1} x^s \qquad (7.3.6)$$

The following recursion systems obtain between the quantities

$$ {}_r\Phi_0(x) = x^{-r} \int_0^x t^{r-1} \, {}_{r-1}\Phi_0(t) \, dt \qquad r = 1, 2, \ldots, q+1 \qquad (7.3.7)$$

$$ {}_1\mathcal{U}_0^{(1)} = x^{-1} \left(u_0^{(0)} - c_0 \right) - {}_1 u_0^{(0)} \qquad (7.3.8)$$

and thereafter

$$ {}_1\mathcal{U}_0^{(s+1)} = x^{-1} (s+1)^{-1} \left\{ u_0^{(s)} - x(2s+1)_1 u_0^{(s)} - x_1 u_0^{(s-1)} \right\} \qquad (7.3.9)$$

and for the functions ${}_r\mathcal{U}_0^{(s)}$ $r = 2, 3, \ldots$

$$ {}_r\mathcal{U}_0^{(s+1)} = (s+1)^{-1} \left\{ {}_{r-1}\mathcal{U}_0^{(s)} - (s+r)_r \mathcal{U}_0^{(s)} \right\} \qquad (7.3.10)$$

For negative r

$$ {}_{-1}\mathcal{U}_0^{(0)} = x^{-1} \, \mathcal{U}_0^{(0)} , \quad {}_{-r}\mathcal{U}_0^{(s)} = (s+1) x^{-1} {}_{-r+1}\mathcal{U}_0^{(s+1)} \qquad (7.3.11)$$

For both systems of transformations (7-2-3) and (7-3-2) the functions ${}_r u_m^{(s)}$ and ${}_r \mathrm{U}_m^{(s)}$ may be built up from ${}_r u_0^{(s)}$ and ${}_r \mathrm{U}_0^{(s)}$ as described in section 2.

If the function $\Phi_o(x)$ satisfies a linear differential equation, further recursion systems for the functions ${}_r u_m^{(s)}$ and ${}_r \mathrm{U}_m^{(s)}$ may be derived as in section 2, and again the inquiry is greatly facilitated if $\Phi_o(x)$ is a hypergeometric function.

7-3. Example : The initial members of the sequence v_m $m = 0,1, \dots$ derived from equation (5-3-22) (which relates, it will be recalled, to the comparison of Wilson's integral with Dawson's integral) are as follows :

n	0	1	2	3	4
v_n	1.075819	1.034540	1.016440	1.010675	1.000794

It will be noted that the limit of this sequence (as is easily verified by use of equation (5-3-7)) is unity, and that the sequence $(v_n - 1)/n$ $n = 1,2, \dots$ is approximately constant. Accordingly the series (5-3-5) is transformed in the following manner : The terms in the series

$$1 - x + x^2 \frac{1}{\frac{3}{2}} - x^3 \cdot \frac{1}{\frac{3}{2} \cdot \frac{5}{2}} + \dots = e^{-x} \int_0^{\sqrt{x}} e^{t^2}\, dt \qquad (7\text{-}3\text{-}1)$$

are subtracted from those of the original series. The resultant series (starting with the second term) is now to be transformed, and the fundamental series is taken to be

$$\frac{1}{1} - \frac{x}{2}\,\frac{1}{\frac{3}{2}} + \frac{x^3}{3} \cdot \frac{1}{\frac{3}{2} \cdot \frac{5}{2}} - \dots = 2h^{-2} \int_0^h e^{-\omega^2}\, d\omega \int_0^\omega e^{t^2}\, dt \qquad (7\text{-}3\text{-}2)$$

where

$$2h^2 = x \qquad (7\text{-}3\text{-}3)$$

Numerical values of (7-3-2) may be extracted from tables in [8] The functions ${}_1 u_0^{(s)}$ are computed from

$$ {}_1u_0^{(1)} = \left\{ u_0^{(0)} - 1 - x\,{}_1u_0^{(0)} \right\} x^{-1} \tag{7.3.4} $$

$$ {}_1u_0^{(2)} = \frac{1}{2}\left\{ u_0^{(1)} x^{-1} - {}_1u_0^{(0)} - 3\,{}_1u_0^{(1)} \right\} \tag{7.3.5} $$

and, since $\Phi_0(x)$ satisfies the differential equation

$$ x y'' + \left(\tfrac{1}{2} - x\right) y' - y = 0 \tag{7.3.6} $$

and the functions $\Phi_0(x)$ and ${}_1\Phi_0(x)$ are related by

$$ x^2\,{}_1\Phi_0'(x) + x\,{}_1\Phi_0(x) + 1 = \Phi_0(x) \tag{7.3.7} $$

there follows

$$ {}_1u_0^{(3)} = \frac{1}{6}\left\{ 1 - (11 - 2x)\,{}_1u_0^{(2)} - \left(\tfrac{11}{2} - 4x\right){}_1u_0^{(1)} - {}_1u_0^{(0)}\left(\tfrac{1}{2} - 2x\right) \right\} \tag{7.3.8} $$

and

$$ {}_1u_0^{(s+4)} = \frac{x(s+1)\,{}_1u_0^{(s)}}{(s+4)(s+3)} - \frac{\left\{6 - 10x + s\left(\frac{33}{2} - 7x + s(17 - 3x + 3s - 3)\right)\right\}}{(s+4)(s+3)(s+2)}\,{}_1u_0^{(s+1)} $$
$$ - \frac{\left\{\frac{33}{2} - 7x + s(14 - 3x + 3s)\right\}}{(s+4)(s+3)}\,{}_1u_0^{(s+2)} - \frac{\left\{3s + \frac{17}{2} - x\right\}}{(s+4)}\,{}_1u_0^{(s+3)} \qquad s = 0, 1, \dots \tag{7.3.9} $$

The quantities ${}_1v_n = (v_{n+1} - 1)/(n+1)$ $n = 0, 1, \dots$ may be computed from the recursion

$$ {}_1v_n = \frac{(n + 1 + \alpha^2)}{n}\,{}_1v_{n-1} - \frac{\alpha^2\,(n + 1/2)}{n(n-1)}\,{}_1v_{n-2} - \frac{\alpha^2}{2n} \qquad n = 2, 3, \dots \tag{7-3-10} $$

If the result of the ensuing Euler-Gudermann transformation is Θ' then the value of the original sum may be recovered as

$$ v_o - 1 + \Phi_o(x) - x\Theta' \tag{7-3-11} $$

In the event, when $\alpha = 0.0625$, $\beta = 2.8865$ ($i.e. x = 11.56$) and

$$ \Theta' \equiv {}_1u_o^{(0)}\ {}_1v_o $$

the numerical value of expression (7-3-11) is -0.126, so that the effect of using a second order transformation is quite striking in this case.

8. Variants of the Euler-Gudermann Tranformation.

8-1. Two further transformations which are akin to that discussed in section 2 and which may easily by derived by formal operational methods are :

8-2. (due to Euler and Gudermann [4]).

$$c_0 v_0 + c_1 v_1 x^{(1)} + c_2 v_2 x^{(2)} + \dots$$
$$= \Phi(x) v_0 + \binom{x}{1} \Delta\Phi(x-1)\Delta v_0 + \binom{x}{2}\Delta^2\Phi(x-2)\Delta^2 v_0 + \dots \qquad (8.2.1)$$

where $$x^{(s)} = x(x-1)\dots(x-s+1) \qquad (8.2.2)$$

and $$\Phi(x) = \sum_{s=0}^{\infty} c_s x^{(s)} \qquad (8.2.3)$$

The derivation is as follows : the operators $\underset{2}{E}$, $\underset{2}{\Delta}$ operate only upon the quantities v_s, whilst $\underset{1}{E}$, $\underset{1}{\Delta}$ operate only on $\Phi(x)$.

Then

$$c_s = \frac{\underset{2}{\Delta}^s \Phi(0)}{s!} \qquad s = 0, 1, \dots \qquad (8.2.4)$$

thus

$$\sum_{s=0}^{\infty} c_s v_s x^{(s)} = (1 + \underset{1}{\Delta}\underset{2}{E})^x v_0 \Phi(0)$$
$$= (1 + \underset{1}{\Delta} + \underset{1}{\Delta}\underset{2}{\Delta})^x v_0 \Phi(0)$$
$$= \underset{1}{E}^x (1 + \underset{1}{\Delta}\underset{1}{E}^{-1}\underset{2}{\Delta})^x v_0 \Phi(0)$$
$$= \sum_{s=0}^{\infty} \binom{x}{s} \underset{1}{\Delta}^s \Phi(x-s) \underset{2}{\Delta}^s v_0 \qquad (8.2.5)$$

8.3.

$$c_0 v_0 + c_1 v_1 x^{(-1)} + c_2 v_2 x^{(-2)} + \dots$$
$$= \Phi(x) v_0 + \binom{x}{1}\Delta\Phi(x)\Delta v_0 + \binom{x+1}{2}\Delta^2\Phi(x)\Delta^2 v_0 + \dots \qquad (8.3.1)$$

where

$$x^{(-s)} = x(x+1)\dots(x+s-1) \qquad (8\text{-}3\text{-}2)$$

and

$$\Phi(x) = \sum_{s=0}^{\infty} c_s\, x^{(-s)} \qquad (8\text{-}3\text{-}3)$$

Assuming again that $\underset{2}{E}$, $\underset{2}{\nabla}$ operate only upon v_s and $\underset{1}{E}$, $\underset{1}{\nabla}$ only upon $\Phi(x)$ there follows

$$\begin{aligned}\sum_{s=0}^{\infty} c_s v_s x^{(-s)} &= (1-\underset{1}{\nabla}\underset{2}{E})^{-x} v_0 \Phi(0) \\ &= \underset{1}{E}^{x}(1+\underset{2}{\Delta}-\underset{2}{\Delta}\underset{1}{E})^{-x} v_0 \Phi(0) \\ &= (1-\underset{1}{\Delta}\underset{2}{\Delta})^{-x}\Phi(x) v_0 \\ &= \sum_{s=0}^{\infty}\binom{x+s-1}{s}\Delta^s\Phi(x)\Delta^s v_0 \end{aligned} \qquad (8.3.4)$$

8-4. A third transformation given by Cherry [12], which has certain points of similarly with the Euler-Gudermann transformation is

$$\sum_{s=0}^{\infty} c_s\, v\,(s)\, x^s = \sum_{s=0}^{\infty} \delta^s\, \Phi\,(x)\, D^s\, v\,(o)\,/\,s! \qquad (8\text{-}4\text{-}1)$$

where

$$\Phi\,(x) = \sum_{s=0}^{\infty} c_s\, x^s \qquad (8\text{-}4\text{-}2)$$

and

$$\delta \equiv x\frac{d}{dx}$$

(In the event Cherry gives a generalised and truncated form of (8-4-1) with a description of estimating the remainder for details of which reference may be made to his paper.)

It may be formally demonstrated as follows

$$\begin{aligned}\sum_{s=0}^{\infty} c_s v(s) x^s &= \Phi(xE)v(0) \\ &= \Phi(xe^{D})v(0) \qquad (D, E \text{ operating on } v(s)) \\ &= \sum_{s=0}^{\infty}\lim_{D\to 0}\left(\frac{\partial}{\partial D}\right)^s \Phi(xe^{D})\frac{D^s}{s!}v(0) \\ &= \sum_{s=0}^{\infty}\lim_{D\to 0}\left(x\frac{\partial}{\partial x}\right)^s \Phi(xe^{D})\frac{D^s}{s!}v(0) \\ &= \sum_{s=0}^{\infty}\delta^s\Phi(x)D^s v(0)/s! \end{aligned} \qquad (8.4.3)$$

Since this transformation requires an explicit formula or at least a differential equation for $v(s)$ to be known, rather than just numerical values, it belongs more to the province of Analysis than Numerical Analysis, and is not pursued further here.

9. Integral Transforms of the Euler-Gudermann Transformation.

9-1. The scope of the Euler-Gudermann transformation may considerably be extended by multiplying equation (2-1-3) throughout by a weight function $\omega(x)$ and integrating the result with respect to x along a suitable contour C. The final result then reads

$$\sum_{s=0}^{\infty} c_s v_s \int_C x^s \omega(x)\, dx \sim \sum_{s=0}^{\infty} \frac{\Delta^s v_0}{s!} \int_C x^s \overset{(s)}{\Phi}(x)\, \omega(x)\, dx \qquad (9\text{-}1\text{-}1)$$

If the functions $\frac{x^s}{s!} \overset{(s)}{\Phi}(x)$ satisfy a linear recurrence relation in s it is by no means certain that the functions $\int_C \frac{x^s}{s!} \overset{(s)}{\Phi}(x)\, \omega(x)\, dx$ satisfy a similar relationship, and it would appear that each case must be judged individually upon its merits.

Some results, relating to a particularly simple choice of $\omega(x)$ and C, may however be stated.

Firstly, if

$$\Phi(x) \sim x^{-1} \int_0^{\infty} e^{-x^{-1} t}\, S(t)\, dt \qquad (9\text{-}1\text{-}2)$$

then

$$S(t) \sim \sum_{s=0}^{\infty} c_s (s!)^{-1} t^s \qquad (9\text{-}1\text{-}3)$$

This result will be of use in a later section in which a convergence theory of the Euler-Gudermann transformation, based upon the properties of $S(t)$, will be given. In particular, if

$$S(t) = (1+t)^{-1} \qquad (9\text{-}1\text{-}4)$$

the result is effectively the Laplace transform of the Euler transformation. In this case

$$\Phi(x) = -z e^z \mathrm{E}i(-z) \qquad z = x^{-1} \qquad (9\text{-}1\text{-}5)$$

and the transformation may be recognised as a particular case $a = b = 1$ of those discussed in section 6.

The Euler transformation may be integrated to yield further useful results. Using the Pochammer double loop integral for the Beta function

$$B(x,y') = \frac{\Gamma(x)\Gamma(y')}{\Gamma(x+y')} = \left\{(1-e^{2\pi i x})(1-e^{2\pi i y'})\right\}^{-1} \int^{(1+),(0+),(1-),(0-)} t^{x-1}(1-t)^{y'-x} dt \qquad (9.1.6)$$

there follows:

$$\frac{1}{y-x}\left\{v_0 + \frac{x}{x-y+1}v_1 + \frac{x(x+1)}{(x-y+1)(x-y+2)}v_2 + \dots\right\}$$

$$= \frac{v_0}{y} - \frac{x}{y(y+1)}\Delta v_0 + \frac{x(x+1)}{y(y+1)(y+2)}\Delta^2 v_0 - \dots \qquad (9.1.7)$$

This may be recognised as a particular case of (8-3-1) in which

$$\Phi(x) = \frac{1}{y-x} \sim \frac{1}{y} + \frac{x}{y(y+1)} + \frac{x(x+1)}{y(y+1)(y+2)} + \dots \qquad (9\text{-}1\text{-}8)$$

For this transformation the quantities $u_0^{(s)}$ and $u_m^{(0)}$ are generated by the recursions

$$u_0^{(0)} = y^{-1}, \quad u_0^{(s)} = -\frac{(x+s-1)}{(y+s)}u_0^{(s-1)} \qquad s=1,2,\dots \qquad (9.1.9)$$

$$u_m^{(0)} = u_{m-1}^{(0)} - \frac{x(x+1)\dots(x+m-1)}{(x-y+1)(x-y+2)\dots(x-y+m)} \qquad m=1,2,\dots \quad (9.1.10)$$

The more general result (8-3-1) may be derived from (9-1-8) by multiplying throughout by $\frac{1}{2\pi_i}\Phi(x+y)$ and integrating the result along the Schläfli contour $(-\infty, 0+, +\infty)$ in the y-plane. The result (8-2-1) follows by reversing the sign of y and the disposition of the Schläfli loop, and carrying out the same procedure.

The important facility which is available when deriving further results from the Euler transformation is that an estimation of the remainder term may feasibly be obtained, based upon the formulae

$$(1-x)^{-1} = \sum_{s=0}^{n} x^s + R_n \qquad (9\text{-}1\text{-}11)$$

$$R_n = x^{n+1}(1-x)^{-1} \qquad (9\text{-}1\text{-}12)$$

this is the case with formulae (9-1-5) and (9-1-7).

10. Convergence Theory.

10-1. The first task of this section will be to derive an expression for $\overset{(p)}{\mathrm{R}}_m$ where

$$\sum_{s=0}^{\infty} c_s v_s x^s \sim \sum_{s=0}^{m-1} c_s v_s x^s + \sum_{s=0}^{p-1} x^{m+s} \overset{(s)}{\Phi}_m (x) \Delta^s v_m + \overset{(p)}{\mathrm{R}}_m \tag{10-1-1}$$

For the purposes of convenience we assume that the representation

$$v_s = \frac{1}{2\pi i} \int_{\mathrm{L}} \frac{v(t)}{t-s}\, dt \qquad s = 0,1, \ldots \tag{10-1-2}$$

exists, and that the rearrangement of terms in the following work is valid. There follows

$$\sum_{s=0}^{\infty} c_s v_s x^s \sim \sum_{s=0}^{m-1} c_s v_s x^s + \sum_{s=0}^{\infty} \frac{c_{m+s} x^{m+s}}{2\pi i} \int_L \frac{v(t)\,dt}{t-m-s}$$

$$\sim \sum_{s=0}^{m-1} c_s v_s x^s +$$

$$+ \sum_{s=0}^{p-1} \frac{c_{m+s} x^{m+s}}{2\pi i} \int_L v(t) \left\{ \frac{1}{t-m} + \frac{s}{(t-m)(t-m-1)} + \cdots + \frac{s!}{(t-m)(t-m-1)\ldots(t-m-s)} \right\} dt$$

$$+ \sum_{s=0}^{\infty} \frac{c_{m+p+s} x^{m+p+s}}{2\pi i} \int_L v(t) \left\{ \frac{1}{t-m} + \frac{p-s}{(t-m)(t-m-1)} + \cdots + \frac{(p+s)(p+s-1)\ldots(s+1)}{(t-m)(t-m-1)\ldots(t-m-p-s)} \right\} dt$$

$$\sim \sum_{s=0}^{m-1} c_s v_s x^s + \sum_{s=0}^{p-1} x^{m+s} \phi_m^{(s)}(x) \Delta^s v_m + R_m^{(p)} \tag{10.1.3}$$

where:

$$R_m^{(p)} \sim \sum_{s=p}^{\infty} c_{m+s} x^{m+s} \left\{ \binom{s-1}{p-1} \Delta^p v_m + \binom{s-2}{p-1} \Delta^p v_{m+1} + \cdots + \Delta^p v_{m+s-p} \right\} \tag{10.1.4}$$

10-2. Cherry's Theory [12].

The result (10-1-4) is due to Cherry; his use of it to discuss the convergence behaviour of the Euler-Gudermann transformation will be summarised here.

I) If $\overset{(p)}{R}_m$ is a real oscillating sequence for $p = \bar{p}, \bar{p} + 1, \ldots$ then quantitive bounds can be given for $\Theta(x)$.

II) If

$$\overline{R}_m^{(p)} = \max_{s \geq m} |\Delta^p v_m| \tag{10.2.1}$$

then:

$$p! \, |R_m^{(p)}| \leq \overline{R}_m^{(p)} \sum_{s=p}^{\infty} s(s-1)\ldots(s-p+1) |c_{m+s} x^{m+s}| \tag{10.2.2}$$

III) If $\Delta^p v_s$ decreases monotonically to zero as $s = m, m + 1, \ldots$ (it may be possible to deduce this information from the difference equation (2-5-6)) then

$$\left\{ \binom{s-1}{p-1} \Delta^p v_m + \binom{s-2}{p-1} \Delta^p v_{m+1} + \ldots + \Delta^p v_{m+s-p} \right\} \Big/ \binom{s}{p}$$

(10-2-3)

decreases as s increases; if t and c_{m+s} are real and positive, then

$$0 < p! \, \overset{(p)}{R}_m < x^{m+p} \, \overset{(p)}{\Phi}_m(x) \, \Delta^p v_m$$

or

$$0 < \overset{(p)}{R}_m < \overset{(p)}{u}_m \, \Delta^p v_m \tag{10-2-4}$$

Example : This result obtains for the transformation discussed in § 4-2.

IV) A further bound for the remainder term may be derived by making the following assumptions :

a) that $\sum_{s=0}^{\infty} c_s x^s = \Phi_o(x)$ has a convergence radius of unity;

b) that $v(z)$ (where $v_s = v(s)$ $s = 0,1, \ldots$) is real for positive real z, is regular when $\mathrm{Re}(z) > 0$, and as $|z|$ tends to infinity has the asymptotic expansion

$$v(z) = Az^{\alpha} + A_1 z^{\alpha_1} + A_2 z^{\alpha_2} + \ldots \tag{10-2-5}$$

uniformly for $|\arg(z)| \leq \pi/2$ and $\alpha - \alpha_1$, $\alpha - \alpha_2$, ... are all not less than a certain positive constant;

c) that on certain arcs of $|x| = 1$, $\Phi_o(x)$ is regular and $\sum_{s=0}^{\infty} c_s s^{\alpha} x^s$ converges.

b) and *c*) imply that $\sum_{s=0}^{\infty} c_s x^s v(s)$ converges on the said arcs of $|x| = 1$; *b*) implies that if $\alpha < 0$, $v(z)$ may be represented by the absolutely convergent integral

$$v(z) = \int_0^\infty e^{-zs} h(s) \, ds \tag{10-2-6}$$

if z is real and positive, and there exists the representation

$$\Delta^p v(z) = \int_0^\infty e^{-zs} h_p(s) \, ds \tag{10-2-7}$$

Thus, using (10-1-4) and assuming $|x| < 1$

$$R_m^{(p)} = p x^m \int_0^\infty e^{-ms} h_p(s) \, ds \int_0^1 (1-t)^{p-1} dt \sum_{r=p}^\infty \binom{r}{p} c_{m+r} x^r (1-t+te^{-s})^{r-p}$$

$$= x^{m+p} \int_0^\infty e^{-ms} h_p(s) \, ds \int_0^1 (1-t)^{p-1} dt \; \Phi_m^{(p)}(x - xt + txe^{-s}) \Big/ (p-1)! \tag{10.2.8}$$

An immediate result which may be deduced from (10-2-8) is that if

$$E_m^{(p)} = \max_{x'(0 \leqslant x' \leqslant x)} \left| \Phi_m^{(p)}(x') \right| \tag{10.2.9}$$

and:

$$\eta_m^{(p)} = \frac{\int_0^\infty e^{-ms} |h_p(s)| \, ds}{\int_0^\infty e^{-ms} h_p(s) \, ds} \tag{10.2.10}$$

then:

$$R_m^{(p)} \leqslant \eta_m^{(p)} E_m^{(p)} \left| x^m \Delta^p v_m \right| \Big/ p! \tag{10.2.11}$$

This result may be applied to the transformation of Mestel's integral (cf. equation (3-4-1)) as discussed in this paper, when $\lambda < 1.0$.

10-3. Van Wijngaarden's Theory [11].

In the ensuing work attention is directed not to the transformation

$$\sum_{s=0}^\infty c_s v_s x^s \sim \sum_{s=0}^\infty x^s \Phi^{(s)}(x) \Delta^s v_0 \tag{10-3-1}$$

but to the integral transform

$$\sum_{s=0}^{\infty} \int_c c_s v_s (t-a)^s e(t,z)\, dt \sim \sum_{s=0}^{\infty} \int_c (t-a)^s \overset{(s)}{S}(t-a)\, \Delta^s v_o\, e(t,z)\, dt \tag{10-3-2}$$

where

$$S(t-a) \sim \sum_{s=0}^{\infty} \frac{c_s (t-a)^s}{s!} \tag{10-3-3}$$

It may at first seem a little artificial to introduce the process of integration into the transformation, but the following convergence theory embraces the possibility that both sides of equation (10-3-1) diverge, and it is by regarding these series as divergent expansions of an integral that they may be given meaning.

If necessary the convergence theory developed in this section, involving properties of $S(t)$, may be transmuted to the consideration of $\Phi_o(x)$ by use of the result

$$\sum_{s=0}^{\infty} c_s x^s \sim \sum_{s=0}^{\infty} z \int_0^{\infty} s(t)\, e^{-zt}\, dt, \quad z = x^{-1} \tag{10-3-4}$$

In particular, if

$$\Phi_o(x) = {}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p;\ \varrho_1, \varrho_2, \ldots, \varrho_q;\ x) \tag{10-3-5}$$

then $$S(t) = {}_pF_{q+1}(\alpha_1, \alpha_2, \ldots, \alpha_p;\ \varrho_1, \varrho_2, \ldots, \varrho_q, 1;\ t) \tag{10-3-6}$$

The following development is due to van Wijngaarden, a few additions and generalisations are made but the theory is essentially his.

Initially the following assumptions are made

a) $S(t-a)$ is regular upon C, and $S^{(s)}(a) \neq 0$ $s = 0,1,\ldots$

b) There exists a region D of the z-plane in which

$$\int_c (t-a)^s e(z,t) < \infty, \quad \int_c (t-a)^s S^{(s)}(t-a) e(z,t)\, dt < \infty \quad s = 0,1,\ldots \tag{10-3-7}$$

c) and a subregion D* of D in which

$$\int_c (t-a)^s |e(z,t)|\, dt < \infty, \quad \int_c (t-a)^s S^{(s)}(t-a) |e(z,t)|\, dt < \infty \quad s = 0,1,\ldots \tag{10-3-8}$$

The fundamental series is taken to be

$$s(z) \sim \sum_{s=0}^{\infty} \frac{\overset{(s)}{S}(a)}{s!} \int_c (t-a)^s e(z,t)\, dt \tag{10-3-9}$$

and the series to be transformed

$$f(z) \sim \sum_{s=0}^{\infty} \frac{\overset{(s)}{S}(a)\, v_s}{s!} \int_c (t-a)^s\, e\,(z,t)\, dt \qquad (10\text{-}3\text{-}10)$$

van Wijngaarden's fundamental result is

Theorem 1 :

If the series

$$U\,(t) = \sum_{s=0}^{\infty} \left| \frac{(t-a)^s\, \overset{(s)}{S}(t-a)\, \Delta^s\, v_0}{s!} \right| \qquad (10\text{-}3\text{-}11)$$

converges uniformly in $|\,t\,| \leqq \delta$ where part of C lies in this circle, then for all z in D for which $\int_c U(t)\,|e(z,t)|\,dt < \infty$ the series

$$\sum_{s=0}^{\infty} \Delta^s\, v_0 \int_c \frac{(t-a)^s\, \overset{(s)}{S}(t-a)\, e\,(z,t)\, dt}{s!} \qquad (10\text{-}3\text{-}12)$$

converges to $f(z)$.

The proof proceeds firstly by noting that under the stated conditions the series (10-3-12) is equal to $\int_c v(t)\,e(z,t)\,dt$ where

$$v\,(t) = \sum_{s=0}^{\infty} \frac{(t-a)^s\, \overset{(s)}{S}(t-a)\, \Delta^s\, v_0}{s!} \qquad (10\text{-}3\text{-}13)$$

converges uniformly for all t in $|\,t-a\,| \leqq \delta$.

Secondly it is shown that

$$\lim_{t \to a} \left(\frac{d}{dt}\right)^m \left\{ \frac{(t-a)^s\, \overset{(s)}{S}(t-a)}{s!} \right\} = \left\{ \begin{array}{ll} 0 & 0 \leqq m < s \\ 1 & m \geqq s \end{array} \right. \qquad (10\text{-}3\text{-}14)$$

so that

$$\begin{aligned} \overset{(m)}{v}(a) &= \overset{(m)}{S}(a) \sum_{s=0}^{m} \binom{m}{s} \Delta^s\, v_0 \qquad (10\text{-}3\text{-}15) \\ &= \overset{(m)}{S}(a)\, v_m \end{aligned}$$

Finally, since $(t-a)^s S^{(s)}(t-a)\Delta^s v_0/s\,!$ and $v(t)$ are analytic in $|t| \leqq \delta$ the series (10-3-12) is, by virtue of (10-3-15), equal to $f(z)$.

The main problem then is that of establishing the convergence of (10-3-11). Since

$$\left| \frac{(t-a)^s \overset{(s)}{S}(t-a)\, \Delta^s v_0}{s!} \right| = \left| \frac{(t-a)^s \overset{(s)}{S}(t-a)}{s!} \right| \left| \Delta^s v_0 \right| \qquad (10\text{-}3\text{-}16)$$

and both constituents of the right hand side of this inequality in many practical applications satisfy linear difference equations it may be possible to deduce the required information from these equations. It will be recalled that if, in the difference equation

$$\sum_{n=0}^{k} \left(\sum_{u=0}^{\mu_n} a_{n,\,u}\, s^u \right) x\,(s+u) = 0 \qquad (10\text{-}3\text{-}17)$$

$\mu_o = \mu_k \geqq \mu_1, \mu_2, \ldots, \mu_{k-1}$ then $x(s)$ has an asymptotic development of the form

$$x\,(s) \sim \sum_{u=0}^{k-1} b_u\, \rho_u^{s}\, p_u\,(s, \log(s)) \qquad (10\text{-}3\text{-}18)$$

where the quantities ρ_s $s = 0,1, \ldots, k-1$ are roots of the indicial equation

$$\sum_{n=0}^{k} a_{n,\,\mu_o}\, \rho^n = 0 \qquad (10\text{-}3\text{-}19)$$

and $p_u(s, \log(s))$ is a polynomial expression in s and $\log(s)$.

If the quantities Δ^s_0, $(t-a)^s S^{(s)}(t-a)/s\,!$ both satisfy difference equations of the form (10-3-17) and the largest root of the indicial equation corresponding to the first is $\rho_1(t)$ and that corresponding to the second is $\rho^*_1(t)$, then the series (10-3-14) converges for all t such that

$$\rho_1(t)\rho^*_1(t) < 1 \qquad (10\text{-}3\text{-}20)$$

An alternative approach to the problem of obtaining a majorant series for (10-3-11) adopted by Van Wijngaarden is his

Theorem 2 :

The following additional assumptions are made :

d) C lies in the inclusion of the circle $|\,t-a\,| \leqq \delta'$ and a half plane formed by a line through a. (Without loss of generality in the ensuing work a will be taken to be the origin and the region in question as the inclusion of $|\,t\,| \leqq \delta'$ and $Re(t) > 0$.)

e) In this region $S(t)$ is analytic and furthermore

$$|S(t)| \leqq A(1 + |t|)^p \tag{10-3-21}$$

where $A, p \geqq 0$.

f)

$$|\Delta^s v_0| \leqq B(1 + s)^q \tag{10-3-22}$$

where $B, q \geqq 0$ (and q may be taken to be an integer).

If *d*), *e*) and *f*) obtain, then for all z in D^* (10-3-12) converges to $f(z)$.

The proof proceeds as follows :

Using a Cauchy integral representation for the s^{th} derivative and *e*) it is shown that

$$\left|\frac{(t+\epsilon)^s S^{(s)}(t+\epsilon)}{s!}\right| \leqslant A\left\{1+2(t^2+\delta'^2)^{\frac{1}{2}}\right\}^p \left\{t(t^2+\delta'^2)^{-\frac{1}{2}}\right\}^k \tag{10.3.23}$$

where $|\epsilon| \leqslant \delta'$

From *f*) it follows that :

$$|\Delta^s v_0| \leqslant (-1)^s q! \binom{-q-1}{s} \tag{10.3.24}$$

or, using (10.3.23) and (10.3.24) :

$$U(t+\epsilon) = \sum_{s=0}^{\infty} \left|\frac{(t+\epsilon)^s S^{(s)}(t+\epsilon)\Delta^s v_0}{s!}\right|$$

$$\leqslant q! AB(2\delta^{-2})^{q+1}\left\{1+2(\delta^2+\delta'^2)^{\frac{1}{2}}\right\}^p \left\{\delta^2+\delta'^2\right\}^{q+1} \tag{10.3.25}$$

so that the series for $U(t)$ converges uniformly for all $|t| \leqq \delta'$.

Further

$$\int_C \left\{1+2(t^2+\delta'^2)^{\frac{1}{2}}\right\}^p (t^2+\delta'^2)^{q+1} |e(z,t)|\, dt$$

$$< (1+2\sqrt{2}\,\delta')^p \int_{C'} (t^2+\delta'^2)^{q+1} |e(z,t)|\, dt$$

$$+ \int_{C''} (1+2\sqrt{2}\, t)^p (t^2+\delta'^2)^{q+1} |e(z,t)|\, dt \tag{10.3.26}$$

where C' is that part of C lying within $|t| \leqq \delta'$ and C'' the remainder of C. But by virtue of *c*) both integrals in (10-3-26) certainly exist.

Thus $\int_C U(t) |e(z,t)|\, dt < \infty$ and the conditions of Theorem 1 are satisfied.

A trivial corollary to Theorem 2 is

Theorem 3 :

If the conditions of theorem 2 are satisfied then the Euler-Gudermann transformations using the line of differences $\Delta^s v_m$ $m = 1,2, \dots$ also converge to $f(z)$.

Remark : All the numerical examples given in this paper, with the exception of those illustrated in Table II and VIII, converge. In these exceptional cases (and they are typical) the Euler-Gudermann sums approach the required result and then diverge. The usefulness of the Euler-Gudermann transformation in these cases therefore depends upon the accuracy required in the final result.

The only remaining difficulty is that of establishing the inequality (10-3-22). This may be made possible by appealing to the difference equation satisfied by $\Delta^s v_0$. Alternatively the following function-theoretic result was given by Van Wijngaarden, Theorem 3.

A necessary and sufficient condition for the inequality (10-3-22) to hold is that if

$$G(t) \sim \sum_{s=0}^{\infty} v_s t^s \qquad (10\text{-}3\text{-}27)$$

then $\Theta(t)$ is analytic and satisfies

$$| G(t) | \leqq C \left(1 - \left| \frac{t}{1+t} \right| \right)^{-r}$$

$$C,\ r \geqq 0\ ,\ \text{for } Re(t) > -1/2. \qquad (10\text{-}3\text{-}28)$$

II. Application of the ε —Algorithm.

11-1. The ε-algorithm provides in certain cases a very powerful technique for the transformation of a slowly convergent sequence S_m $m = 0,1, \dots$.

The ε-algorithm relationships are [13]

$$\overset{(m)}{\varepsilon}_{s+1} = \overset{(m+1)}{\varepsilon}_{s-1} + \frac{1}{\overset{(m+1)}{\varepsilon}_{s} - \overset{(m)}{\varepsilon}_{s}} \quad m,s = 0,1, \dots \qquad (11\text{-}1\text{-}1)$$

$$\overset{(m)}{\varepsilon}_{-1} = 0\ ,\ \overset{(m)}{\varepsilon}_{0} = S_m \qquad m = 0,1, \dots \qquad (11\text{-}1\text{-}2)$$

and it is a fundamental result in the theory of this transformation that if

$$S_m = a + \sum_{s=0}^{k} \lambda_s^m \sum_{h=0}^{\mu_s} a_{s,h}\, m^h \qquad (11\text{-}1\text{-}3)$$

$$\sum_{s=0}^{k} (\mu_s + 1) = n \qquad (11\text{-}1\text{-}4)$$

then

$$\varepsilon_{2n}^{(m)} = a \qquad m = 0,1, \ldots \qquad (11\text{-}1\text{-}5)$$

and it is a matter of numerical experience that if the sequence S_m $m = 0,1, \ldots$ is dominated by a term of the form (11-1-3), then the sequence $\varepsilon_{2s}^{(0)}$ $s = 1,2, \ldots$ converges far more rapidly to a than the sequence S_m $m = 0,1, \ldots$ (which may in the event diverge without affecting the validity of (11-1-5)).

Since it was remarked in the preceding section that in certain cases the sequence $u_s^{(0)} \Delta^s v_0$ $s = 0,1, \ldots$ may well be dominated by a term of the form $\left\{ \rho_1(t) \overset{*}{\rho}_1(t) \right\}^s p_k(s)$ the Euler-Gudermann transformation appears to offer a promising point of application for the ε-algorithm. This is indeed substatiated by numerical experience.

Three examples will be given. In the first the initial values $\varepsilon_0^{(r)}$ $r = 1,2, \ldots$ are provided by the partial sums along the line $m = 4$ of Table VI, relating to the transformation of Wilson's integral by comparison with Dawson's integral.

(Note firstly that the value $\varepsilon_0^{(0)} = 0$ has been inserted, secondly that the results contained in Tables VI and IX, and indeed in all the tables of this paper, were computed using fixed length (12 decimal) floating point arithmetic, but for the purposes of display have been somewhat abbreviated).

In the second example the initial values $\varepsilon_0^{(r)}$ $r = 0,1, \ldots$ are provided by the diagonal $m = 0$ in Table I, relating to the generalised Euler transformation of Mestel's integral; and in the third the initial values are taken from the diagonal $m = 0$ of Table VII relating to the comparison of the integral of Goodwin and Staton with the exponential integral.

The even order columns of the ε-arrays for these examples are displayed in Tables IX, X and XI respectively.

m \ s	0	2	4	6	8	10	12
0	0.0						
1	-674.426088	-337.547987					
2	-1.338442	-0.477994	-0.250266				
3	-0.479093	.168587	.136331	-0.127578			
4	.251002	.136343	.125465	.124179	-0.123942		
5	.174700	.127588	.124179	.123911	.123911	-0.123919	
6	.145572	.125000	.123942	.123911	.123911	.123943	-0.123937
7	.133515	.124220	.123914	.123919	.123943	.123937	-0.123936
8	.128267	.123994	.123920	.123887	.123937	.123936	
9	.125912	.123937	.123926	.123940	-0.123937		
10	.124838	.123928	.123930	-0.123937			
11	.124345	.123930	-0.123926				
12	.124120	-0.123933					
13	-0.124018						

Table IX

m \ s	0	2	4	6	8	10	12
0	0.000						
1	0.091	0.220					
2	.144	.259	0.295				
3	.181	.281	.308	0.318			
4	.208	.295	.315	.323	0.326		
5	.228	.304	.320	.324	.327	0.328	
6	.244	.310	.323	.326	.328	.328	0.328
7	.257	.315	.324	.327	.328	0.328	
8	.268	.318	.326	.328	0.328		
9	.276	.320	.326	0.328			
10	.284	.322	0.228				
11	.290	0.323					
12	0.295						

TABLE X

m \ s	0	2	4	6	8	10	12
0	0.0						
1	0.64043830	0.62188138					
2	.62132764	.57721154	0.60430379				
3	.60799329	.60409910	.60486619	0.60514597			
4	.60497931	.60472503	.60511319	.60515164	0.60514817		
5	.60474481	.60484839	.60514958	.60514436	.60513126	0.60513304	
6	.60493033	.60541280	.60514361	.60509908	.60513299	.60513341	0.60513366
7	.60506432	.60516378	.60513824	.60513256	.60513333	0.60513364	
8	.60512140	.60514499	.60513514	.60513323	0.60513354		
9	.60513809	.60513992	.60513380	0.60513342			
10	.60513974	.60513882	0.60513340				
11	.60513765	0.60504045					
12	0.60513560						

TABLE XI

Acknowledgements :

The formulae relating to the expansion of Wilson's integral in ascending and asymptotic series were taken from the investigation of the integral by F. W. J. Olver and E. T. Goodwin. The author is grateful to them for drawing his attention to this example. The results displayed in the Tables of this paper were computed upon the X1 computer at the Mathematical Centre, using the ALGOL-compiler constructed by E. W. Dijkstra and J. A. Zonneveld.

The list of References to Part II occur at the end of Part I.